RCA

Application is being made for second class mail rates.

Mathematics News Letter

SUBSCRIPTION. $1.00 *Per Year*

Published under auspices of the Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics.

To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 3 BATON ROUGE, LA., JANUARY, 1929 NO. ~~4~~ 5

MATHEMATICS A MENTAL GYMNASIUM

If a biceps, trained with dumbbell and horizontal bar, can have its acquired power applied to running a harrow or pushing a saw, so can the power of mental concentration developed in algebra be applied to working out a cross-word puzzle or detecting a thief or solving a business problem. If a football team can transfer its group energy, quickness and driving force, developed by exercise on sawdust dummies and imaginary opponents, to the field of conflict with actual opponents, so can the skills grown in scientific problem solving in a mathematics class room be transferred to any field in which are problems to be solved.

Every mathematical process implies a mental process. A repeated mathematical process implies a repeated process of the mind. Whether or not a process of mind can be resolved into the activities of separate faculties, as reason, memory imagination, attention, whether or not such separate faculties have even an existence, certain it is that repeated action of any sort means easier action. Repeated thinking means easier thinking. Repeated analysis means more ability to analyze.

Mathematics as a mental gymnasium is unequalled. Its

processes are so closely woven that oft-repeated exercises of the mind are necessary in endless instances in order to insure correct mathematical results. Its symbolism and machinery are of such nature that clearness and sharpness of meaning after a sufficiently prolonged period of training in mathematics become the habitual demand of the mathematical worker. The indefiniteness and vagueness of expression which always go with the over-use of qualifying adjectives have little if any place in a series of mathematical propositions. Such statements as "A is good", or "B is bad" are not of mathematical quality. But "$A>B$, $C<D$, $x=y$" are strictly mathematical. Another factor in the mathematical process which contributes to clearness and definite expression is the practice of brevity in notation. In ordinary language we say: The square on the hypotenuse of the right triangle is equal to the sum of the squares on the two legs of the triangle. In mathematics we say $x^2+y^2=r^2$.

—*S. T. S.*

THE TRANSFER OF ACQUIRED POWER

Much ado has been made over the distinction between conscious and unconscious transfer of acquired mental power. The psychologists appear to have concluded from different experiments made that there is little of unconscious transfer to other fields of the skills acquired in mathematics study. On the other hand we gather that they are agreed that a measure of conscious transfer of acquired skill from one field to another can be made. The fact that there may be but little of unconscious transter from field to field of acquired skill is, to our thinking, of very slight importance, at least insofar as the matter has application to mathematics. The thing of most significance is one's ability to CONSCIOUSLY carry over to field F_2 a power or skill acquired in field F_1. It is the conscious element of transfer which is the wide-awake element and hence the most effective one. That boxer would be doomed who should rely upon unconscious transfer of a hitting strength acquired in punching a bag to the act of knocking out a boxing opponent. The young man at Amherst—mentioned by ex-President Olds—made a very CONSCIOUS endeavor (and succeeded) to transfer to the language field skills acquired in mathematics. —*S. T. S.*

*THE MINIMUM TRAINING PREREQUISITE TO THE SUCCESSFUL PURSUIT OF COLLEGE MATHEMATICAL STUDY

By A. C. MADDOX
Head Math. Dept., Louisiana Normal College

(Part Two)

The habit of generalizing in mathematical work is of exceedingly great importance. The illustration of the triangle problem just given suggests the great value of the student's having learned the general fact that any three independent elements of a triangle determine its shape and size. The generalization that the roots of the equation $ax^2+bx+c=0$ are equal if and only if $b^2-4ac=0$ is necessary to the understanding of quadratic equations in one unknown.

The habit of doing independently as much of the work as possible is of the greatest value in the mathematical training of high school students. If the student acquires the attitude that prompts him to want to play the role of a beginning research worker, the success of his mathematical study is much more probable than if he is satisfied merely to read, even understandingly, the developments made in the textbook or by the teacher. Much of the subject matter of high school mathematics is of such nature that interested students can really discover for themselves the important parts of it. It is worthwhile for the teacher to undertake to develop this type of independence in students by means of such devices as that of asking students to discover and prove every fact and relation they can regarding the parallelogram. In this connection it should be clear that the solution of so-called "originals" is much more valuable in developing independence in students than the study of propositions whose proofs are given in the textbook.

The importance of establishing thoroughly the habit of checking the solutions of all exercises and problems needs very special emphasis. Furthermore much attention should be devoted to the question of the best methods of checking in the various types of exercises and problems. The first thing the student should do upon the completion of his problem computations is habitually to consider the reasonableness of his result,

*Read before the Mathematics Section of L. T. A. Nov. 24, 1928.

else he may fail to note that the discount as computed on a bill of goods is greater than the marked price of the goods or that he has a railway train traveling at the rate of 750 miles per hour or that his computation has revealed the interesting fact that the President of the United States is working for the munificent sum of $2.50 per day.

If the student is to acquire the training that will enable him really to succeed in his more advanced mathematical work, he must establish the habit of abbreviating his solutions at least to some considerable extent. The student who habitually persists in writing in full such solutions as

$$x^2-5x+6=0$$
$$(x-2)\ (x-3)=0$$
$$x-2=0$$
$$x=2$$
$$x-3=0$$
$$x=3$$

will have real difficulty in keeping up with discussions in college classes where such things are frequently stated casually as that the roots of the equation $x^2-x-12=0$ are 4 and -3. The students should be encouraged to develop the ability to abbreviate their solutions. For example, if a student knows that in selling a certain piano for $480, a dealer lost 20% of the cost and if he wishes to find for how much the dealer would have had to sell it in order to gain 20% of the cost, he should be commended and not criticised and penalized if he states that the required selling price is $3/2\times$ $480.

Another very important habit in mathematical work is that of substituting simple numbers for difficult numbers in problems in order to discover more readily the correct procedures in the solutions. If a student is assuming that A can do a certain piece of work in x days and if he desires to express the fractional part of the work A can do in one day, the habit of mentally substituting some small numerical integer for x momentarily is indispensable in case he has difficulty in forming the desired expression.

The few study habits that I have briefly discussed are among the more important ones, but of course there are others that are necessary to adequate training in high school mathematics. The ones mentioned will probably be sufficient for il-

lustrative purposes in this discussion. The interested teacher of mathematics can, by various means, complete the list of essential study habits.

It is quite important that high school teachers keep in mind that these habits are more easily established during the high school period than later, by virtue of the laws of habit formation that suggest that the study habits employed during three and one-half years of high school mathematical study will strongly tend to be in force in connection with his college mathematical study.

As to what particular subject matter of high school mathematics a student must master in order to be able to succeed in his work in college mathematics, it is obviously impossible for me to state here in a detailed list what I believe it to be. Let us undertake, however, to indicate roughly some of the general types of mathematical work that a student must do with reasonable thoroughness in high school if he is to be able to succeed in his more advanced mathematical study.

It will be generally admitted, no doubt, that he must develop the ability to handle with accuracy and with reasonable speed the fundamental operations with numerical integers, common fractions, decimal fractions, and the more familiar denominate numbers. Much of the poor mathematical work of college students is due to their inadequate mastery of arithmetical operations.

A student should be able to solve with facility the three "of", the three "more than" and the three "less than" general types of arithmetical exercises and problems, both in the ordinary fraction language and in "percent" language. A college student who cannot determine quickly and without the use of pencil and paper the correct answer to such a question as "56 is 2/9 less than what number?" or "84 is what percent more than 72?" has little chance of attaining success in real college mathematical study. He should have the well established habit of simplifying such a complex fraction as $\dfrac{2+1/3\cdot\frac{1}{4}}{5/6-\frac{1}{2}}$ merely by multiplying both the numerator and the denominator by the least number that can be thus used in making both the numerator and the denominator integral instead of the habit of simplifying the

numerator and the denominator separately and then dividing the former by the latter.

The student should acquire a thorough understanding of the meaning and the process of cancellation in arithmetic; for few phases of algebra present more and greater difficulties to students than those phases requiring or permitting the rapid use of cancellation; and this topic cannot be mastered without considerable use of arithmetic illustrations.

The general method of finding the square roots of numbers should be thoroughly mastered, and the student should learn the meaning of an irrational number and how to write such numbers and how to reduce them to the standard form given in high school textbooks. A college student is greatly handicapped in certain phases of mathematical work if he does not see almost instantly such things as that $3\sqrt{288}=36\sqrt{2}$. He should be trained in the practice of indicating a series of operations; and he should become thoroughly familiar with the fact that much time is often saved by indicating several operations before performing any one of them; and he should acquire a thorough knowledge of the standard forms of expression for elementary use in indicating arithmetical operations. It is quite rare that a college freshman is found who interprets a series of four or five indicated operations according to accepted practice among textbook writers and advanced students of mathematics. This deficiency is necessarily a great handicap to a student not only in his effort to understand the mathematical language of those who use correct language but also in his effort to express his own thoughts correctly to others.

He should know the real meanings of multiplication and division so well and should be so thoroughly trained in their accurate applications that he will not use such phrases and statements as 4 dollars times 5, multiply 8 times 7, $12 \div 3 =$ four pencils, 2 ft. times 3 ft.$=$6 sq. ft., 18 ft.$\div 3 = 6$ yd., and 12 cts. $\div 3$ ct.$=$4 pencils.

I know that there are some who regard such things as trivial matters and who contend that it is not serious if a student persists in committing such errors, but in this opinion I find myself simply unable to concur, and particularly when there is involved the problem of trying to build upon such a foundation a training in college mathematics that serves a really worthwhile purpose.

The thing that is of fundamental importance in the successful study of algebra is the fact that algebraic expressions are numbers and not mere letter combinations. A few days ago a freshman in one of my classes pointed to the written statement "$ax+bx=(a+b)x$" and asked, "which one of these x's is this?" This general type of deficiency is often revealed in connection with work in other mathematical subjects. One of my students recently asked regarding the formula $\sin 2x=2 \sin x \cos x$, "Aren't there two cos x's as well as two sin x's?"

But in connection with the importance that the student regard algebraic expressions as numbers, it must be made clear to the student that with relatively rare exceptions the fundamental operations with literal numbers cannot be completely performed. In this connection, previously acquired training in indicating operations with numerical quantities is of great value. For a student to learn in a purely mechanical manner that $(a+b)(a-b)=a^2-b^2$, is of doubtful value; but if he makes this general fact meaningful by observing that $32 \cdot 28=900-4$, that $47 \cdot 53=2500-9$, and that the product of any two positive integers whose difference is an even integer may be found by subtracting from the square of the median integer the square of half the difference between the given integers, then the fact takes on a much greater significance and hereby becomes really educative for him.

Although operations with literal numbers cannot usually be completely performed, the student must develop the ability to perform with accuracy and with reasonable speed whatever of any series of fundamental indicated operations can be performed, in order that he may write any given expression in the most desirable or most usable form. This implies of course, that he must be able to handle with facility the fundamental operations with algebraic expressions.

The student must develop the ability to solve readily any first degree equation in one unknown and to make use of such equations in the solution of concrete problems. Likewise he must develop the ability to solve systems of first degree equations in two or three unkowns and to use such systems in the solutions of concrete problems that require their use.

(To be concluded in the February News Letter)

EDUCATIONAL THEORY REDUCED TO MATHEMATICAL FORM

By W. PAUL WEBBER
Louisiana State University

(First Paper)

It has been said that every science tends to become mathematical, and, in finished form, actually does become virtually a chapter in mathematical science. It is a matter of observation that the great sciences that contribute to our comfort and welfare have become largely mathematical. Owing to the volume of testimony as to the undesirability of requiring mathematics in the curriculum one would not suspect that educational theory itself could be corralled into the mathematical camp. To show that this would be easy, even if not immediately probable, witness the following as a possible introductory chapter:

1. **Definition.** Any thing that enters into consciousness is an element of state of mind.

Principles Relating to Mind

2. State of mind depends upon:
   (a) Immediate environmental conditions.
   (b) History of the individual (his experience.)

3. Every state of mind tends to arouse action. Action may be (a) Advantageous or
   (b) Non advantageous.

4. Repetition of an act under the same or similar conditions tends to make the act habitual under those conditions.

5. The kind of action performed at any time is determined by the state of mind immediately preceding it.

6. Habit established with respect to any action in a given situation influences state of mind when the same or similar circumstances recur.

7. Habit of action under given conditions makes that action more probable when similar circumstances recur than a possible new action. That is, habit exercises a conservative influence in respect to choice of action.

8. Habit is necessary to efficient action in a given situation.

9. Life presents a variety of situations calling for efficient action by the individual.

10. **Definition.** Education is a process (or set of processes) of forming habits of efficiency in

(a) Observation (use of the sense organs).
(b) Thinking (attending and arranging the elements of state of mind).
(c) Body movements.

Theorem I. Education is possible.
This follows from (2), (6), (7).

Theorem II. Variety of habits is necessary to education.
This follows from (7), (8), (9).

Theorem III. Education may be suitably guided.
This follows from (1), (2), (3), (4), (5), (6), (7).

THE DEVELOPMENT OF ELEMENTARY GEOMETRY

By H. E. BUCHANAN
Professor of Mathematics, Tulane University

1. **Introduction.** The beginnings of geometry were made by the Babylonians and Egyptians. The former divided the circle into 360 parts and introduced our present method of counting time and measuring angles. The latter introduced methods of computing areas. In Egypt the regular overflow of the Nile undoubtedly forced men to learn something of lines, angles and areas in order that they might re-locate each man's land when the water had subsided. Indeed the Egyptian word for geometer means a rope stretcher. The oldest known manuscript, containing any geometry, is the papyrus of Ahmes, deciphered in 1871 by Eisenlohr. In it formulas for the area of a triangle, parallelogram and trapezoid are given. Their formulas were inaccurate, for example, the area of a triangle was given as half the product of one side and the base. This is approximately true for a triangle which is nearly a right triangle. Historians say that Ahmes wrote his treatise about 1700 B. C.

The geometry set forth by Ahmes is purely empirical, that is, there were no axioms, theorems or corrollaries. They merely stated results. Ahmes was a priest. The Priesthood had complete control of all such matters as science, literature and religion. It is interesting to note that another treatise written about 400 or 500 A. D. shows no improvement over that of

Ahmes 2000 years older. Some have drawn the conclusion that it will ever be thus when scientific affairs fall into the hands of the clergy since their science soon becomes sacred and to change it becomes heresy.

I do not know whether this is true or not, for those who are not connected with any religious creed cling tenaciously to the old. For example, in 1821 there appeared in Germany an arithmetic, written for use in the schools, in which only the Roman notation was used!

The early Hebrews and Bablyonians used 3 for π, the ratio of the circumference to the diameter of a circle. The reference to the Hebrew value is I Kings, VII, 23 where Solomon's temple is described. "And he made a molten sea ten cubits from one brim to the other, and it was round all about—and a line thirty cubits did compass it." The Egyptians had a slightly more accurate value of π, viz, 3.1604. This value is not given explicitly by Ahmes but it is implied in his rule for finding the area of a circle. It is interesting, and of value, to the modern youth, to know that the Babiyonians applied geometry in the study of Astronomy and the Egyptians used it in the vast engineering projects of building the pyramids before the time of Abraham.

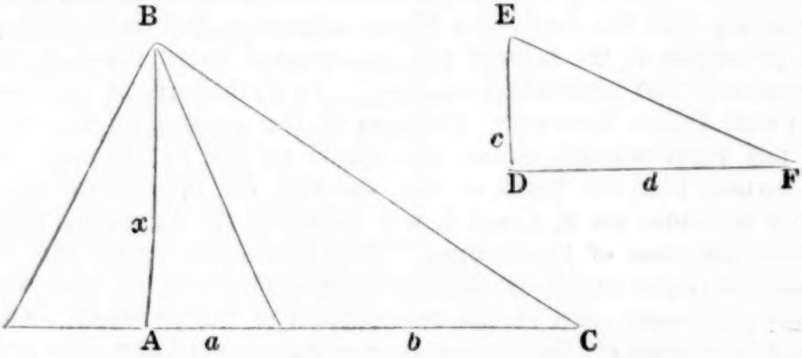
2. **Thales, 640, 546 B. C.** But the Egyptians knew nothing of logical development, nothing of hypothesis and conclusion. It remained for the Greeks, the greatest thinkers of antiquity, to make this development. Thales of Miletus, (Circ 640, 540 B. C.) one of the seven wise men, began it. He was a merchant and became interested in geometry while visiting Egypt. Plutarch declares that Thales soon excelled the Priests, and amazed King Amasis by measuring the height of the pyramids from their shadows.

This was probably done as follows: The two right triangles A B C and D E F are similar. The distances a, b, c and d can measured. Hence x may be found from the proportion

$$\frac{x}{a+b} = \frac{c}{d}$$

To Thales proofs of the following theorems are ascribed. Vertical angles are equal. The base angles of an isosceles triangle are equal. Any circle is bisected by every diameter. Two triangles are congruent if two angles and the included side are

equal, respectively. He must have known the theorem that corresponding sides of similar triangles are proportional for he used it to find the height of the pyramids.



Other theorems were doubtless known to Thales but the contribution he made was not so much the theorems he proved as the method he introduced of reasoning from hypothesis to conclusion instead of the empirical process of the Egyptians.

Thales founded a school known as the Ionic school. It was very different from our present day colleges. He gained much fame by predicting an eclipse of the sun, the first one ever predicted. He presents also the first example of the absent-minded college professor. It is said that while he was taking a walk one evening he became so absorbed in studying the stars that he fell into a ditch. An old woman helped him out and after scolding him roundly said, "How can you know what is going on in the heavens when you can't see what is at your own feet?"

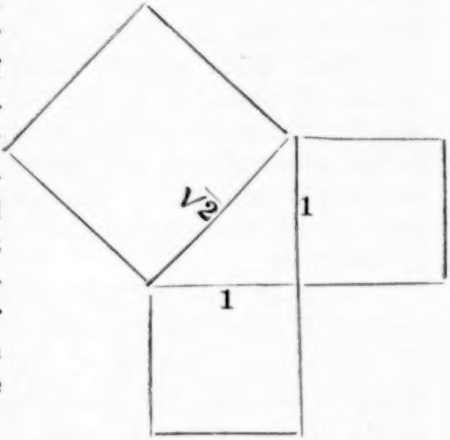
3. **Pythagoras, 580, 500 B. C.** There probably were other scholars and teachers in the time of Thales about whom we know very little or nothing. My reason for thinking this is that progress in any science has never been made by isolated men, but always in association with other men thinking along like lines, with one man, usually, as the central figure of a certain period. Next central figure of Greek geometry, chronologically, was Pythagoras. He is said to have studied in Egypt and to have traveled over the then known world. He must have been familiar with the results of Thales being about two generations, 60 years, later. He founded a school at Croton in Southern Italy. His followers bound themselves into a brotherhood with

rites approaching those of the Masonic order. Its members were forbidden to give out any of their discoveries.

Pythagoras, like Thales, wrote no books or treatises. The Eudemean summary says of him that he "changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated its theorems in an immaterial and intellectual manner." To Pythagoras is ascribed the well known theorem: The sum of the squares on the legs of any right triangle equals the square on the hypotenuse. It is certain that the truth of this theorem for the special case, when the sides are 3, 4, and 5, was known to the Egyptians long before the time of Pythagoras. They undoubtedly used this to construct right angles in their land surveying and in their engineering works such as the construction of the pyramids. Often I have seen my father construct a right angle with 6, 8 and 10 links of a Gunthers chain, in the same manner as the Egyptian priests did it 5000 years ago! Pythagoras proved that this property holds for all right triangles of any shape whatever. He based his proof on certain postulates which were then accepted by scholars. We are told that he was so jubilant that he scarificed a hundred oxen to the Muses who inspired him. His proof has not been handed down to us. We can only conjecture as to its nature. The theorem is now known as the theorem of Pythagoras. In the centuries which have elapsed since the time of Pythagoras it has been proved in many ways. In the American Mathematical Monthly, Vols. III to VI., 1896 to 1899, over 100 different proofs are given.

During the middle ages this theorem received several nicknames. In the European Universities it was known as Magister Matheseos because the masters' examination did not, usually, extend beyond this point. The name Pons Asinorum, the bridge of fools, has been applied to it. Undoubtedly this was an appropriate name from the student's view point when we take into account the difficulty of proving it. It was sometimes called the Windmill in England from the similarity of the figure to a Dutch Windmill. Another of the discoveries of the Pythagoreans is that of the irrational numbers. It is arithmetic in character but it came out of the theorem of Pythagoras. If each of the legs be taken as 1 then the hypotenuse must be $\sqrt{2}$. It is easy to show that it cannot be a whole number nor the

quotient of two whole numbers. It was a new kind of a number that could not be the result of counting nor the result of dividing a whole number into any number of parts. The Pythagoreans regarded it as a symbol of the unspeakable. The first one of their number who divulged it to a person outside their brotherhood is said to have been shipwrecked so angry were the Gods at his lack of reverence.



The Pythagoreans proved many other theorems. Among them are: The sum of the angles of a triangle is 180°. The plane about a point is completely filled by 4 squares, 6 equilateral triangles, 3 regular hexagons. They established a number of theorems on areas.

4. **Plato.** Soon after the death of Pythagoras, the brotherhood, of which he was the guiding genius, was dissolved and his followers scattered. The center of intellectual supremacy passed over from southern Italy to Athens. There the Pythagorean practice of secrecy ceased to be observed. Th Platonic school did much to bind together the works of Thales and Pythagoras. They examined carefully the foundations on which the theory was built. Antiphon and Bryson introduced the theory of limits. The philosophers came in with their contribution to the development of geometry. Plato himself was not a professed mathematician but he suggested improvements in the definitions and the logic. He placed a very high estimate on the educational value of geometry and inscribed over the entrance to his school the legend: "Let no one unacquainted with geometry enter here." This is the first instance, so far as I know, of entrance requirements. Let the deans note that one of their chief functions is an ancient and honorable one.

By the time of Plato there was an extensive literature on geometry. And as is always the case certain irregular fellows were introducing unorthodox methods. For example, machines were made to construct certain curves needed in the trisection

of an angle, the duplication of the cube and other problems in which the time honored ruler and compasses did not seem to be sufficient. Plato drew a sharp distinction between these two schools of geometry by the famous "Plato's Restriction." This was his dictum that only the ruler and compass could be used in the constructions of elementary geometry. The ruler must not be graduated, i. e., distances could be transferred from one part of the figure to another only by use of compass and not by the ruler.

We shall refer to this a little later in connection with one method of trisecting an angle which we shall give.

5. **Euclid. 300 B. C.** After Athens was conquered by Philip of Macedon her power declined and the center of learning passed to the newly founded City of Alexandria. The central figure of the Alexandrian school was Euclid who flourished about 300 B. C. He was not so great as an original investigator as he was as writer and teacher. He took the results obtained by other investigators and wove them into a rigorous text book. His Elements of Geometry was used as a text book in secondary schools of England till 1900. It was only driven out of American schools by Wentworth. One of my earliest memories is the hearing of Euclid and "Pons Asinorum" discussed around the family fireside while the older children were studying their geometry with my father's help. The sacred writings excepted, no book has been so widely read or variously translated as Euclid. Even today all geometry texts have "books" instead of chapters like other texts. This arose from the fact that the texts were written on parchment which were rolled into a double roll, one unrolling while the other rolled up, the reader reading between. These rolls were called books. They corresponded to our chapters.

Very little is known of the life of Euclid. He was said to be gentle and kind to any one who could in the least degree advance mathematical science. When Ptolemy, King of Egypt, asked him if there was not some short cut, easy way to learn geometry, shorter than studying through the "Elements," Euclid replied, "There is no Royal road to geometry." A youth once asked him "What do I gain by learning those things." This is a very common question today. Scarcely a day goes by but that I get the same question either asked or implied. If it were not for the fact that I would encourage the questioners too much

I should answer as Euclid did. He called his slave and said "Give him three pence since he must gain out of what he learns."

This reply was characteristic of the idealism of the Greeks. They developed geometry not for its utility but for its beauty and its adaptability to a perfect system of logic. A notable example of this is in their study of the Conic Sections. They knew the ellipse, parabola and hyperbola and most of their properties. They obtained these as exercises in solid geometry without the remotest idea so far as we can tell, that there ever would be any use for them. Now they are in use in Astronomy, Physics, Engineering, Ballistics and in almost every field of applied science. This has been true again and again in mathematics.

6. **Archimedes, 287, 212 B. C.** The greatest mathematician of all the ancients was Archimedes. He was born in Syracuse in Sicily. Cicero tells us he was of low birth. He laid the foundations of modern engineering and was famous for his many mechanical inventions, particularly his war engines, but he himself prized his purely mathematical studies more. His contributions were in solid geometry mainly and on the sphere and cylinder. To him we owe our formulas $4\pi r^2$ for the surface, and $4/3\pi r^3$ for the volume of the sphere.

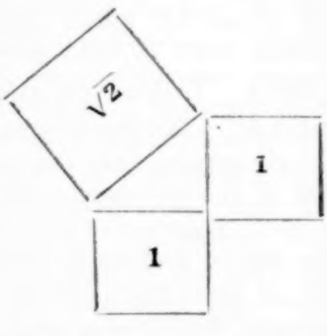
He proved that the surface and volume of a sphere are 2/3 that of the circumscribing cylinder. He was particularly useful to King Hieron during the siege of Syracuse by the Romans. The legend that he set fire to the Roman ships with a powerful lens is probably a fiction. When Syracuse was taken by the Romans Archimedes was studying some geometrical diagram drawn in the sand. To the Roman soldiers who found him he cried "Noli turbari meas circulos", but the soldier, being insulted, killed him. The Roman general Marcellus appreciated his genius and erected a tomb in his honor bearing the figure of a sphere inscribed in a cylinder. Cicero records finding this tomb covered with rubbish.

The work of Archimedes paved the way for Appollonius of Perga who completed the work mentioned above as Conics. With them the golden age of Greek discovery came to an end. The Romans who reached the zenith of their power soon after this were not even good imitators. A science of geometry with definitions, axioms and proofs was wholly unknown in Rome. At the time of Julius Caesar the approximate formulas of the Egyp-

tians were still in use and as late as 500 A. D. Boethius wrote a copy of Euclid's geometry with all the proofs omitted!

7. **Three Famous Problems.** From the time of Pythagoras there was a great interest on the part of many mathematicians in the areas affected by Greek culture in three famous problems. These three problems have profoundly influenced the development of mathematics and hence have had a profound influence on the development of our own civilization. When historians quit being hypnotized by battles, generals and the glamor of victory, these three problems will stand out as having more effect than the wars of any age. They are: To square a circle, to duplicate the cube, and to trisect an (every) angle.

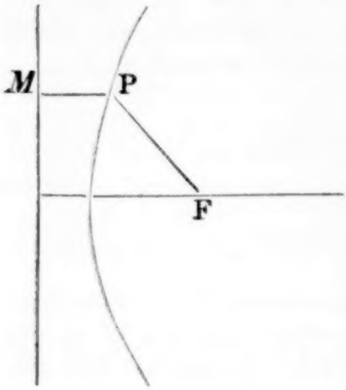
The first one means to find a square equal in area to a circle, and originated in the efforts to find the area of a circle. The second doubtless was an attempt to extend to solid geometry the obvious result of the isosceles right triangle which clearly enables us to construct a square twice another. Eratosthenes gives another origin for it. The Delians once suffered from a plague and were ordered by the oracle to double a certain cubical altar. Thoughtless workmen doubled the edges, thereby multiplying the volume by eight, which did not appease the Gods. Plato was appealed to but long before the problem was solved the plague had died out. Neither of these problems can be solved by the use of ruler and compass only. We discuss in greater detail the third problem, which often, even today, breaks into the newspapers.



The Greeks early learned how to bisect any angle with the ruler and compass. They soon learned too how to trisect a right angle and any of its integral multiples or parts. They were stopped however by the problem of the trisection of any angle so long as they restricted themselves to the use of the ruler and compass only. Nearly every Greek geometer tried to solve this problem.

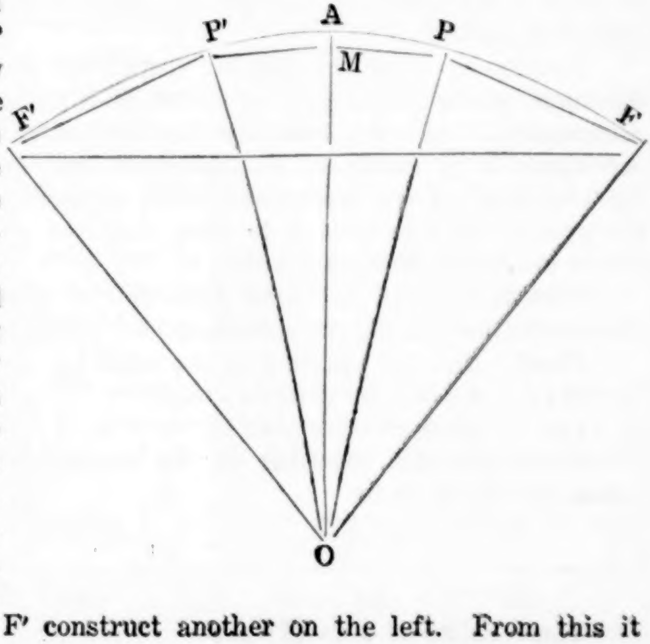
Indeed they solved it by the use of the curves but none of them succeeded by the use of circles and straight lines. It

was not till the nineteenth century that a definite proof was given that the problem could not be solved under Plato's Restrictions, that is, with the use of any ungraduated ruler and compass. Whenever you see in the newspapers a statement that some budding mathematician, a Sidis for example, has achieved immortal fame by trisecting an angle you may at once put it down as bunk. It can't be done with a ruler and compass and it has been done half a dozen different ways more than 2000 years ago by the Greeks themselves using special curves.



One way of trisecting an angle is by means of an hyperbola of eccentricity 2. This means that the curve is such that the distance of every point on it from a fixed point F is just twice the distance from a fixed line.

That is, for any point P on the curve, F P =2 M P. Now let F O F′ be the angle to be trisected. We first draw the circle with center at O, the chord F′ F, and A O bisecting F O F′. Now using O A as the fixed line and F as the fixed point, construct an hyperbola with eccentricity 2.



Similarly with F′ construct another on the left. From this it

is evident that P′ M P=P F= F′ P′ and hence if P and P′ are joined to O the angle at O will be trisected. One must not confuse this with the trisection of the angle with ruler and compass, for the hyperbola cannot be drawn without a specially constructed machine. Hippias of Elis did even better. He invented a curve, the quadratrix, by the use of which he could cut any angle into any number of equal parts. For the description of this curve and other curves invented by the Greeks the reader is referred to standard works on the history of mathematics.

PLOTTING THE CUBIC

By IRBY C. NICHOLS
Louisana State University

The present paper offers two suggestions which should be time-saving both to high school and college teachers of mathematics in their plotting of cubic equations. The author has used them with profit a good many years and, although he does not recall ever having seen them treated in any text on algebra or calculus, he feels that they possess merit sufficient to warrant special attention.

Familiarity with the popular definitions of maximum and minimum points and a point of inflection is assumed. It is also assumed that we can obtain the first derivative of a cubic and can apply it in obtaining its maximum and minimum points. No knowledge of the second derivative is required: the point of the present brief treatise is to show that *the second derivative is not needed in plotting a cubic.*

THEOREM I. *In the cubic equation, its point of inflection bisects the line joining its maximum and minimum points.*

Proof: Let the equation of the cubic be

(1) $y=ax^3+bx^2+cx+d$; then

(2) $y'=3ax^2+2bx^1+c$.

From (2) the abscissas of the maximum and minimum points are found to be

$$x_1=-\frac{b}{3a}+\frac{\sqrt{b^2-3ac}}{3a}, \text{ and } x_2=-\frac{b}{3a}-\frac{\sqrt{b^2-3ac}}{3a}$$

These values of x in (1) give

$$y_1=f(x_1)=\left\{\frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right\}-\left\{\frac{2b^2+5ac}{9a}\right\}\frac{\sqrt{b^2-3ac}}{3a};\text{ and}$$

$$y_2=f(x_2)=\left\{\frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right\}+\left\{\frac{2b^2+5ac}{9a}\right\}\frac{\sqrt{b^2-3ac}}{3a}.$$

Whence the co-ordinates of the mid-point (x_m, y_m) of the line joining the maximum and minimum points are

$$(x_m, y_m)=\left\{\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right\}=\left\{-\frac{b}{2a}, \frac{2b^3}{27a^2}-\frac{bc}{3a}+d\right\}$$

But these are precisely the co-ordinates of the point of inflection. For from (2) above we have $y''=6ax+2b$, from which the abscissa of the point of inflection is found to be $x_i=-\frac{b}{3a}$; then from (1), $y_i=f(x_i)=f(-\frac{b}{3a})=\frac{2b^3}{27a^2}-\frac{bc}{3a}+d$. Therefore $(x_m, y_m)=(x_i, y_i)$.

Hence the conclusion: *To obtain the point of inflection of a cubic, obtain its maximum and minimum points, and then find the mid-point of a straight line joining them.*

REMARKS. It will be obvious, of course, which point is the maximum point and which is the minimum point. We notice also that the co-ordinates of the point of inflection must always be real and rational if the co-efficients a, b, c and d are real and rational.

The point of inflection of the cubic, once obtained, is of special value in its further plotting, because of the truth of a second theorem, which is more general and of which Theorem I is a corollory.

THEOREM II. *The point of inflection of a cubic is its point of symmetry: that is, any straight line cutting the cubic in a point of inflection* I (x_i, y_i) *will cut it in two other points P and Q such that PI=IQ.*

Proof: By Theorem I, we found the co-ordinates of the point of inflection $(x_i\ y_i)$ to be $(-\frac{b}{3a}, \frac{2b^3}{27a^2}-\frac{bc}{3a}+d)$. Mov-

ing the origin of co-ordinates to this point, the equation of the cubic becomes $y=ax^3+(c-\frac{b^2}{3a})x$, from which are missing both the constant term and a term containing the second power of x. Quite clearly any straight line $y=mx$ through the new origin I will cut the cubic in this point and in two others points P and Q equidistant from it, thus making PI=IQ. Hence our second theorem.

Corollary 1. Formulae for obtaining the co-ordinates of the three points of intersection of a straight line through the point of inflection of the cubic may be obtained by simulating $y=mx$ with $y=ax^3+(c-\frac{b^2}{3a})x$, giving I $=(0, 0)$,

$$P=\left[\frac{1}{a}\sqrt{\frac{b^2-3ac+3am}{3}}, \frac{m}{a}\sqrt{\frac{b^2-3ac+3am}{3}}\right] \text{ and}$$

$$Q=\left[-\frac{1}{a}\sqrt{\frac{b^2-3ac+3am}{3}}, -\frac{m}{a}\sqrt{\frac{b^2-3ac+3am}{3}}\right]$$

the point of inflection being the origin of co-ordinates. In practice, however, it is more economical to obtain co-ordinates of points in the ordinary way by chosing values of x and then, from the given equation, finding the corresponding values of y. The teacher will use inspection and synthetic division where advantageous, of course.

Corollary 2. By giving to m various special values, formulae that are simple may be obtained for finding P and Q. In particular, if $m=0$, we have the intercepts for the new x—axis.

Finally then, to plot the cubic, first, find its maximum and minimum points; second, find the co-ordinates of the mid-point of a straight line joining these points, which mid-point will be hereafter known as the point of inflection. The formulae $-\frac{b}{3a}$, $f(-\frac{b}{3a})$ may be employed at the outset, or later as a check.

Third, find two or three other points, say P_0, P_1, P_2 and, with the point of inflection I as a center of symmetry, find Q_0, Q_1, Q_2, the symmetrical points of P_0, P_1, P_2 respectively. This data is sufficient for plotting a nice graph of the given cubic, in which graph will be accurately located the three important points, the maximum and minimum points and the point of inflection.

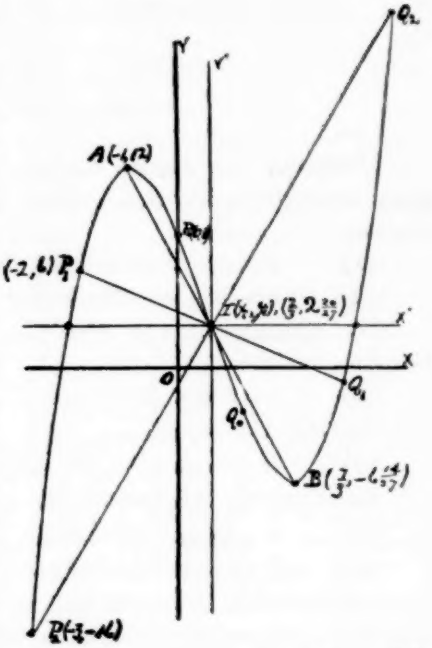


Illustration. In the accompanying figure, we have the graph of $y=x^3-2x^2-7x+8$. Its first derivative is $y'=3x^2-4x-7$. From this first derivative and the given equation, we find the maximum and minimum points to be $A(-1, 12)$ and $B(\frac{7}{3}, -6\frac{14}{27})$. Obviously A is the maximum point and B the minimum point. By Theorem I, the point of inflection is found to be $I=(x_i, y_i)=(\frac{2}{3}, 2\frac{20}{27})$. The y intercept is 8 obviously. Hence the co-ordinates of P_0 are (0, 8). Arbitrarily taking the abscissas of P_1 and P_2 to be —2 and —3 respectively, we find the corresponding ordinates to be 6 and —16 respectively. Hence the co-ordinates of P_1 are (—2, 6), and of P_2(—3, —16). Plotting the points A, B, I, P_0, P_1, P_2 and, also, the points Q_0, Q_1, Q_2 symmetrical to P_0, P_1, P_2 respectively with I as a center of symmetry, we then sketch in the corresponding curve. This curve is the graph of our given equation. Its maximum point, its minimum point, and its point of inflection, its three most important points, are accurately plotted; the second derivative has not been mentioned; and practical use has been made of the fact that, in the cubic, its points of inflection is its center of symmetry.

SOME DEDUCTIONS FROM THE COSINE LAW

By S. T. SANDERS
Louisiana State University

Denoting the angles and corresponding opposite sides of a plane triangle by A, B, C, and a, b, c, we have from the Law of Cosines,

(1) $c^2=a^2+b^2-2ab \cos C$

(2) $a^2=b^2+c^2-2bc \cos A$

By addition of (1) and (2) with proper transposition and division, we have

(3) $b=c \cos A+a \cos C$

Solving (1) for b,

(4) $b=a \cos C+\sqrt{c^2-a^2 \sin^2 C}$

Eliminating b between (3) and (4),

$$c \cos A=\sqrt{c^2-a^2 \sin^2 C}$$

or, $c^2(1-\sin^2 A)=c^2-a^2 \sin^2 C$

(5) or, $c \sin A=a \sin C$

Thus, the elimination of a small letter between two expressions of the Law of Cosines involving five distinct letters results in the Law of Sines.

Again, (3) may be written,

$$1=\frac{c}{b}\cos A+\frac{a}{b}\cos C$$

or, using Law of Sines,

$$1=\frac{\sin C \cos A}{\sin B}+\frac{\sin A \cos C}{\sin B}$$

or, since $\sin B=\sin (C+A)$,

$$1=\frac{\sin C \cos A+\sin A \cos C}{\sin (A+C)}$$

(6) that is, $\sin (A+C)=\sin C \cos A+\cos C \sin A$

Once more, (3) may be written,

$$b=-c \cos (B+C)+a \cos C$$

whence, $\cos (B+C)=\frac{a}{c}\cos C-\frac{b}{c}$

$$=\frac{\sin A\cos C}{\sin C}-\frac{\sin B}{\sin C}$$

$$=\frac{\sin (B+C)\cos C}{\sin C}-\frac{\sin B}{\sin C}$$

$$=\frac{\sin B\cos^2 C+\cos B\cos C\sin C-\sin B}{\sin C}$$

$$=\frac{\sin C\cos B\cos C-\sin B\sin^2 C}{\sin C}$$

(7) or, $\cos (B+C)=\cos B\cos C-\sin B\sin C$

By making use of the relation $B=90°-B'$, in case B is acute, (6) and (7) may be made to yield,

(8) $\cos (A-B')=\cos A\cos B'+\sin A\sin B'$

and

(9) $\sin (A-B')=\sin A\cos B'-\cos A\sin B'$

While the validity of these formulae as developed is restricted by the condition $A+B+C=180°$, it is well known that (6), (7), (8), (9) are true for all values of the angles involved.

Thus is there a sense in which we say that out of the Law of Cosines may be developed analytical trigonometry.

NOTE ON A CLASS OF CURIOUS LOCI

By H. L. SMITH
Louisiana State University

In a recent issue of this NEWS LETTER (Vol. III, No. 2, Oct., 1928) the writer published a note in which occurred the locus of an equation of the form $x^2+(|y|-k)^2=r^2$. This is an instance of a general class of loci, concerning which it is possible to state the following

THEOREM. If $f(xy)=0$ is the equation of a locus L, then (1) $f(|x|, y)=0$ is the equation of the locus which consists of those points of L which are in the first and fourth quadrants together with their reflections in the y-axis; (2) $f(x, |y|)=0$ is the equation of the locus which consists of the points of L which

are in the first and second quadrants together with their reflections in the x-axis; (3) $f(|x|, |y|)=0$ is the equation of the locus which consists of the points of L which are in the first quadrant together with their reflections in the origin, their reflections in the x-axis, and their reflections in the y-axis.

The proof of (3) only will be given; the proofs of (1) and (2) are similar. In the first quadrant, $|x|=x$, $|y|=y$ and the locus of $f(|x|, |y|)=0$ is identical with that of $f(x,y)=0$. In the second quadrant, $|x|=-x$, $|y|=y$ and the locus of $f(|x|, |y|)=0$ is identical with that of $f(-x, y)=0$, which is the reflection in the y-axis of L. In the third quadrant, $|x|=-x$, $|y|=-y$ and the locus of $f(|x|, |y|)=0$ is identical with that of $f(-x, -y)=0$, which is the reflection in the origin of that of $f(x, y)=0$. In the fourth quadrant, $|x|=x$, $|y|=-y$ and the locus of $f(|x|, |y|)=0$, is identical with that of $f(x, -y)=0$, which is the reflection in the x-axis of that of, $f(x,y)=0$. This proves (3).

We append some examples.

EXAMPLE 1. $x-|y|=0$.

The locus of this equation, by (2), consists of two half-lines meeting at (0, 0) and bisecting the first and fourth quadrants, respectively.

EXAMPLE 2. $|x|-|y|=0$.

The locus of this equation is, by (3), the pair of straight lines through the origin with slopes 1, —1 respectively.

EXAMPLE 3. $|x|+|y|=1$.

The locus of this equation is, by (3), the square whose vertices are (1, 0), (0, 1), (—1, 0), (0, —1).

EXAMPLE 4. $x^2+(|y|-2)^2=1$.

The locus of this is, by (2), the pair of circles each of unit radius and having their centres at (0, 2), (0, —2) respectively.

EXAMPLE 5. $x^2+(|y|-1)^2=1$.

The locus of this is, by (2), the pair of circles of unit radus each, which are tangent to the x-axis at (0, 0).

EXAMPLE 6. $x^2+(|y|-1)^2=4$.

The locus of this is, by (2), the figure which consists of the points of the circle $x^2+(y-1)^2=4$ which lie in the first and second quadrants together with their reflections in the x-axis.

EXAMPLE 7. $(|x|-h)^2+(|y|-k)^2=r^2$.

Let the reader discuss the locus of this equation under various hypotheses as to the relative magnitudes of h, k, r.